The Importance of Being Correlated: Implications of Dependence in Joint Spectral Inference across Multiple Networks

Konstantinos Pantazis

kpantaz1@jhu.edu

Applied Mathematics & Statistics (AMS)

Johns Hopkins University

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Inference across Multiple Networks

Often, data consist of a collection of networks with aligned vertices:

- Multilayer networks
- Time-varying networks
- Multiple samples of networks



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 - Embedding networks into a Euclidean space allows us to utilize tools from traditional statistics.
 - 2 The independence assumption rarely holds in real-data applications.
- The main goal is to bring awareness to the induced correlation that may arise in such joint network embeddings.

Statistical network inference

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- Consistent estimation of the underlying model parameters
- Asymptotic normality results
- Subsequent inference tasks:
 - 1 2-graph or *M*-graph hypothesis testing
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What are the moving parts in joint network embedding procedures?

- Network model
- 2 Technique to aggregate networks
- 3 Embedding method

Random Dot Product Graph (RDPG)



Each vertex *i* is associated with a **latent position** $X_i \in \mathbb{R}^d$ drawn from a distribution *F*, with support supp $(F) \subset B_d(1)$

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- Given X_i, X_j , the adjacency matrix A of a sampled graph G is given by:

$$A_{ij} = \begin{cases} 1 & \text{,with probability } \langle X_i, X_j \rangle \\ 0 & \text{,otherwise.} \end{cases}$$

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■ Why d-RDPG?

- d-RDPG is an analytically tractable model.
- Yet, encompasses a broad range of random graph models such as positive semidefinite SBM and Erdos-Renyi.

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Adjacency Spectral Embedding (ASE)

The *d*-dimensional adjacency spectral embedding (ASE) of A is obtained by

$$\widehat{X}_A = U_A S_A^{1/2} \in \mathbb{R}^{n \times d}$$

- $\blacksquare \ S_A \in \mathbb{R}^{d \times d} :=$ diagonal matrix whose entries are the top d eigenvalues of $|A| = (A^T A)^{1/2}$
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Under RDPG, ASE **consistently estimates** (up to orthogonal transformation) the data matrix $\mathbf{X} \in \mathbb{R}^{n \times d}$ (Sussman et al, 2014).

OMNIBUS Embedding (Levin et al. 2017)

■ Joint-RDPG **model:** Given data matrix **X**, for each $k \in [m]$,

 $A^{(k)} \sim \text{RDPG}(\mathbf{X})$

Note: $A^{(k)}$'s are assumed independent, and the same data matrix **X** is used to generate all *m* graphs.

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Omnibus matrix:

$$M = \begin{bmatrix} A^{(1)} & \frac{A^{(1)} + A^{(2)}}{2} & \frac{A^{(1)} + A^{(3)}}{2} & \cdots & \frac{A^{(1)} + A^{(m)}}{2} \\ \frac{A^{(2)} + A^{(1)}}{2} & A^{(2)} & \frac{A^{(2)} + A^{(3)}}{2} & \cdots & \frac{A^{(2)} + A^{(m)}}{2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{A^{(m)} + A^{(1)}}{2} & \frac{A^{(m)} + A^{(2)}}{2} & \frac{A^{(m)} + A^{(3)}}{2} & \cdots & A^{(m)} \end{bmatrix} \in \mathbb{R}^{mn \times mn}$$

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OMNI Embedding:

$$\text{OMNI}(A^{(1)}, A^{(2)}, \cdots, A^{(m)}, d) = \text{ASE}(M, d) = U_M S_M^{1/2} \in \mathbb{R}^{mn \times d}$$

Advantages of OMNI embedding

- **1** Consistency and asymptotic normality results when the graphs are sampled from the same distribution
- **2** The omnibus embedding produces *m* distinct estimates for each *vertex*
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Multi-scale inference:

Estimated latent position of $vertex \ 1$ in G_1 Estimated latent position of $vertex \ 2$ in G_1

$$ASE(M, d) = \begin{bmatrix} \hat{X} \\ \hat{Y} \end{bmatrix} =$$

Estimated latent position of $vertex \ n$ in G_1 Estimated latent position of $vertex \ 1$ in G_2 Estimated latent position of $vertex \ 2$ in G_2

Estimated latent position of $vertex \ n$ in G_2

 $\in \mathbb{R}^{2n \times d}$

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CLT (Avanti et al., 2016): Consider $A \sim d\text{-RDPG}(X)$ and $\widehat{X}_A = U_A S_A^{1/2}$. Under mild assumptions, there exists an orthogonal matrix Q such that

$$\sqrt{n}(\widehat{X}_A Q - X)_i \xrightarrow{\mathcal{D}} \mathcal{N}(0, \Sigma(X_i))$$

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correlation $(A_{ij}, B_{ij}) = \rho$ for all i, j.

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• Let $\hat{X}_A = U_A S_A^{1/2}$ and $\hat{X}_B = U_B S_B^{1/2}$ their embeddings. Then, there exist orthogonal matrices Q_1, Q_2

$$\sqrt{n}(\widehat{X}_A Q_1 - \widehat{X}_B Q_2)_i \xrightarrow{\mathcal{D}} \mathcal{N}(0, 2(1-\rho)\Sigma(X_i))$$

OMNI induces "flat" correlation between estimates

CLT for OMNI estimates: Let *M* denote the omnibus matrix. Let $\widehat{\mathbf{X}}_M = U_M S_M^{1/2}$ and denote the estimates from network $A^{(s)}$ as $\widehat{\mathbf{X}}_M^{(s)}$. For fixed indices s_1, s_2 , there exist orthogonal matrix *W*

$$\sqrt{n} \left((\widehat{X}_M^{(s_1)} - \widehat{X}_M^{(s_2)}) W \right)_i \xrightarrow{\mathcal{D}} \mathcal{N}(0, \frac{1}{2} \Sigma(X_i))$$

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 $\rm OMNI$ embedding induces correlation equal to $\rho=0.75$ between estimates across all networks.

Induced correlation in OMNI embedding

Network space Embedded space G_{u} G_{u}

corr(G, H) = 0

 $corr(\widehat{X}_u, \widehat{X}_v) = 0.75$

"Flat" correlation can mask the signal present in a time-series of networks application

Application: Analysis of Aplysia californica escape motor program of Hill et al. (2020)

- 20 min recording of action for 82 neurons
- One minute into the recording, stimulus was applied to nerve 9.

The stimulus results to initial rapid galloping followed by a slower rhythmic crawling

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To extract a network time series from the recording

- Bin the motor program into 24 bins, each \approx 50 second long
- Convert each bin into a weighted matrix
- \blacksquare Embed these 24 matrices using $\rm OMNI$ procedure.



First graph \implies relaxing state. Second graph \implies firing state. Rest of the graphs \implies galloping and crawling states.

OMNI:

- Successfully detects the stimulus in the second graph
- However, the induced flat correlation masks the transition from galloping to crawling and creates an artificial similarity between graphs 1 and some of the graphs k > 2 in the embedded space

Generalized OMNI (genOMNI)

- R-RDPG model: Extend Joint-RDPG model to incorporate latent correlation across networks
- Generalized omnibus matrix: The block entries of the generalized matrix M are convex combinations of A⁽ⁱ⁾'s.

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- CLT for genOMNI estimates:

$$\sqrt{n} \Big((\widehat{X}_{\mathfrak{M}}^{(s_1)} - \widehat{X}_{\mathfrak{M}}^{(s_2)}) W \Big)_i \xrightarrow{\mathcal{D}} \mathcal{N}(0, 2(1 - \rho(s_1, s_2)) \Sigma(X_i)), \text{ where }$$

$$\begin{split} \rho(s_1,s_2) = \underbrace{1 - \frac{\sum_{q=1}^{m} (\alpha(s_1,q) - \alpha(s_2,q))^2}{2m^2}}_{\text{method-induced correlation}} \\ + \underbrace{\frac{\sum_{q < l} \left(\alpha(s_1,q) - \alpha(s_2,q) \right) (\alpha(s_2,l) - \alpha(s_1,l) \right) \rho_{q,l}}{m^2}}_{\text{model-inherent correlation}} \end{split}$$

and $\alpha(k,q)$ is the total weight put on $A^{(q)}$ in the k-th block-row of \mathfrak{M} .

Real data experiment cont'd



Dampened OMNI:

$$\mathfrak{M}_{damp}^{(k,\ell)} = \begin{cases} \frac{A^{(k)} + \ell A^{(\ell)}}{\ell + 1} & \text{ if } k < \ell, \\ A^{(k)} & \text{ if } k = \ell \end{cases}$$

- Detects the stimulus in the second graph
- Distinguishes the relaxing state from the other states.
- Captures (imperfectly) the transition from galloping (graphs 3, 4) to crawling (graphs 5-24).
- Picks out an unstable dynamic (graphs 13-24) not apparent from simple visual inspection of the firing traces.

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Summary

- Identify and analyze the *phenomenon of induced correlation*, which is an artifice in joint network embeddings.
- First to show theoretical guarantees (consistency, asymptotic normality) under a correlated multiple network model.
- Extend previous methodology to a *family of models* making genOMNI suitable for meaningful subsequent inference, especially for *time series of networks* applications.

Future work: corr2omni algorithm

Given a correlation structure for a collection of networks, choose weights in the genOMNI setting that would reproduce (approximately) this structure in the embedded space.



corr(G, H) = 0.7

 $corr(\widehat{X}_w, \widehat{X}_v) \approx 0.7$

Future work

Identify and analyze the induced correlation in other joint embedding procedures (e.g., COSIE-MASE, Arroyo et al. 2020).



Thank You !

Paper: https://www.jmlr.org/papers/v23/20-944.html
Contact: kpantaz1@jhu.edu