Implications of Dependence in Joint Spectral Inference across Multiple Networks

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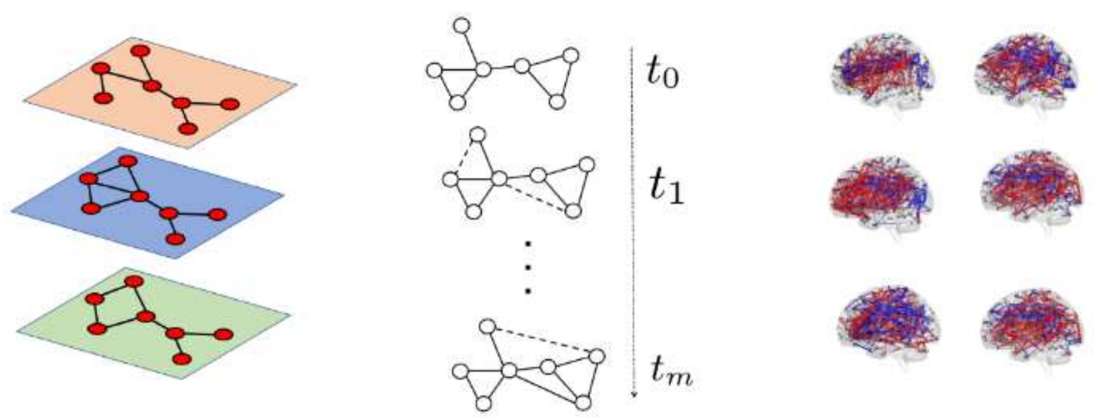
NEURAL INFORMATION PROCESSING SYSTEMS

Introduction

- Joint spectral inference on multiple networks is a subfield of Machine Learning and comprises of methods which *simultaneously embed* a collection of networks into a common low-dimensional space.
- Previous joint embedding procedures assume independence across networks which rarely holds in real-world applications.



Goals

1. Develop a theoretically proven joint embedding procedure which allows for both independent and correlated networks.
2. Discovery and characterization of the phenomenon of induced correlation in joint spectral embeddings.

Background

- ❖ Inference on a single network
  - ➢ Given data matrix **X**,

$$A \sim Bernoulli(XX^T)$$

  - ➢ Sussman et al. proved that

$$\hat{X}_A = U_A S_A^{1/2}$$

  consistently estimates **X**.

- ❖ Inference on multiple networks (OMNIBUS Embedding)
  - ➢ Given data matrix **X**, for $k \in [m]$

$$A^{(k)} \sim Bernoulli(XX^T)$$

  - ➢ Define omnibus matrix *M*

$$M = \begin{bmatrix} A^{(1)} & \frac{A^{(1)}+A^{(2)}}{2} & \frac{A^{(1)}+A^{(3)}}{2} & \cdots & \frac{A^{(1)}+A^{(m)}}{2} \\ \frac{A^{(2)}+A^{(1)}}{2} & A^{(2)} & \frac{A^{(2)}+A^{(3)}}{2} & \cdots & \frac{A^{(2)}+A^{(m)}}{2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{A^{(m)}+A^{(1)}}{2} & \frac{A^{(m)}+A^{(2)}}{2} & \frac{A^{(m)}+A^{(3)}}{2} & \cdots & A^{(m)} \end{bmatrix}$$

  - ➢ Levin et al. proved that each block of

$$\hat{X}_M = U_M S_M^{1/2}$$

  consistently estimates **X**.
  - ➢ Levin et al. also proved asymptotic normality of rows of the OMNIBUS embedding.

Generalized Omnibus Embedding (genOMNI)

- Given data matrix **X**, for $k \in [m]$

$$A^{(k)} \sim Bernoulli(XX^T)$$

  and

$$corr(A^{(k_1)}, A^{(k_2)}) = \rho(k_1, k_2) \in [0,1]$$

- Each block of the generalized omnibus matrix $\widetilde{M}$ is

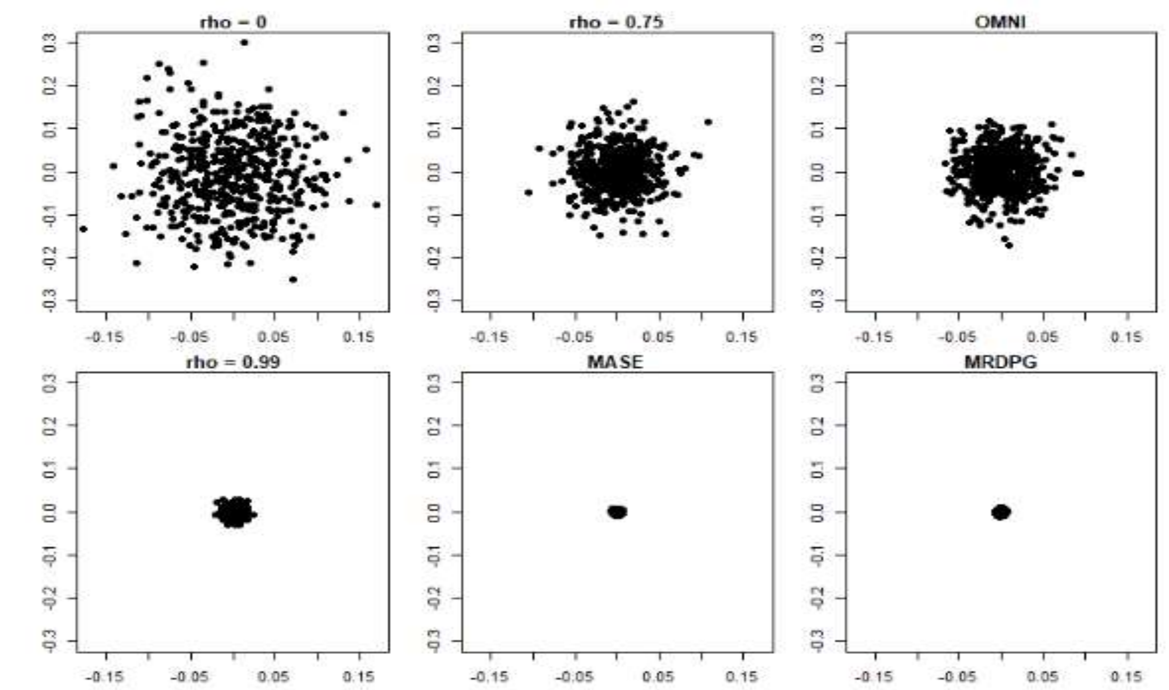
$$\widetilde{M}_{ij} = convex\ combination\ of\ A_1, A_2, \cdots, A_m$$

✓ Each block of $\hat{X}_{\widetilde{M}}$ consistently estimates **X**.
✓ Asymptotic normality of rows of the genOMNI embedding.

Induced correlation phenomenon

- Although in Levin et al. the networks are assumed to be (conditionally) independent, the network estimates in the embedded space are correlated. In fact, the OMNIBUS embedding induces "flat" correlation equal to 0.75 between estimates across all networks.
- We showed that the total correlation between estimates of two networks $A^{(s_1)}$, $A^{(s_2)}$ can be written as

$$\rho(s_1, s_2) = \underbrace{1 - \frac{\sum_{q=1}^{m}(\alpha(s_1,q) - \alpha(s_2,q))^2}{2m^2}}_{\text{method-induced correlation}} + \underbrace{\frac{\sum_{q<l}\Big(\alpha(s_1,q) - \alpha(s_2,q)\Big)\Big(\alpha(s_2,l) - \alpha(s_1,l)\Big)\rho_{q,l}}{m^2}}_{\text{model-inherent correlation}}.$$



The effect of induced correlation in the embedded space by the OMNIBUS, MASE and multi-RDPG joint embeddings in contrast to the inherent correlation ρ.

Summary / Future work

- genOMNI is a quite general, flexible and multi-scale procedure with theoretical guarantees suitable for meaningful subsequent inference on tasks such as *multi-scale change-point detection, multiple-graph hypothesis testing, vertex, graph classification*.
- Characterize the induced correlation in other joint embedding procedures.
- Develop an algorithmic approach in which, given a correlation structure for a collection of networks, choose weights in the genOMNI setting that would reproduce this structure in the embedded space (corr2omni algorithm).

References

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